On Generalizing Rough Set Theory

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Abstract. This paper summarizes various formulations of the standard rough set theory. It demonstrates how those formulations can be adopted to develop different generalized rough set theories. The relationships between rough set theory and other theories are discussed.

1 Formulations of Standard Rough Sets

The theory of rough sets can be developed in at least two different manners, the constructive and algebraic methods [16–20, 25, 29]. The constructive methods define rough set approximation operators using equivalence relations or their induced partitions and subsystems; the algebraic methods treat approximation operators as abstract operators.

1.1 Constructive methods

Suppose U is a finite and nonempty set called the universe. Let $E \subseteq U \times U$ be an equivalence relation on U. The pair $apr = (U, E)$ is called an approximation space [6, 7]. A few definitions of rough set approximations can be given based on different representations of an equivalence relation.

An equivalence relation E can be conveniently represented by a mapping from U to 2^U, where 2^U is the power set of U. More specifically, the mapping $[\cdot]_E$ is given by:

$$[x]_E = \{y \in U \mid xEy\}. \tag{1}$$

The subset $[x]_E$ is the equivalence class containing x. The family of all equivalence classes is commonly known as the quotient set and is denoted by $U/E = \{[x]_E \mid x \in U\}$. It defines a partition of the universe, namely, a family of pairwise disjoint subsets whose union is the universe. A new family of subsets, denoted by $\sigma(U/E)$, can be obtained from U/E by adding the empty set $\emptyset$ and making it closed under set union, which is a subsystem of 2^U. In fact, it is an σ-algebra of subsets of U and a sub-Boolean algebra of the Boolean algebra $(2^U, ^c, \cap, \cup)$. Furthermore, $\sigma(U/E)$ defines uniquely a topological space $(U, \sigma(U/E))$, in which $\sigma(U/E)$ is the family of all open and closed sets [6].

Under the equivalence relation, we only have a coarsened view of the universe. Each equivalence class is considered as a whole granule instead of many individuals [21]. They are considered as the basic or elementary definable, observable, or measurable subsets of the universe [7, 20]. By the construction of

$\sigma(U/E)$, it is also reasonable to assume that all subsets in $\sigma(U/E)$ are definable. To a large extent, the standard rough set theory deals with the approximation of any subset of U in terms of definable subsets in $\sigma(U/E)$.

From the different representations of an equivalence relation, we obtain three constructive definitions of rough set approximations [17, 19, 21, 27]:

Element based definition:

$$\begin{aligned}
\underline{apr}(A) &= \{x \mid x \in U, [x]_E \subseteq A\} \\
&= \{x \mid x \in U, \forall y \in U[xEy \Longrightarrow y \in A]\}, \\
\overline{apr}(A) &= \{x \mid x \in U, [x]_E \cap A \neq \emptyset\} \\
&= \{x \mid x \in U, \exists y \in U[xEy, y \in A]\};
\end{aligned} \tag{2}$$

Granule based definition:

$$\begin{aligned}
\underline{apr}(A) &= \bigcup\{[x]_E \mid [x]_E \in \sigma(U/E), [x]_E \subseteq A\}, \\
\overline{apr}(A) &= \bigcup\{[x]_E \mid [x]_E \in \sigma(U/E), [x]_E \cap A \neq \emptyset\};
\end{aligned} \tag{3}$$

Subsystem based definition:

$$\begin{aligned}
\underline{apr}(A) &= \bigcup\{X \mid X \in \sigma(U/E), X \subseteq A\}, \\
\overline{apr}(A) &= \bigcap\{X \mid X \in \sigma(U/E), A \subseteq X\}.
\end{aligned} \tag{4}$$

The three equivalent definitions offer different interpretations of rough set approximations. According to the element based definition, an element x is in the lower approximation $\underline{apr}(A)$ of a set A if all its equivalent elements are in A, the element is in the upper approximation $\overline{apr}(A)$ if at least one of its equivalent elements is in A. According to the granule based definition, $\underline{apr}(A)$ is the union of equivalence classes which are subsets of A, $\overline{apr}(A)$ is the union of equivalence classes which have a nonempty intersection with A. According to the subsystem based definition, $\underline{apr}(A)$ is the largest definable set in the subsystem $\sigma(U/E)$ that is contained in A, $\overline{apr}(A)$ is the smallest definable subset in $\sigma(U/E)$ that contains the set A.

One may interpret $\underline{apr}, \overline{apr} : 2^U \longrightarrow 2^U$ as two unary set-theoretic operators called approximation operators. The system $(2^U, ^c, \underline{apr}, \overline{apr}, \cap, \cup)$ is called a rough set algebra [16]. It is an extension of the set algebra $(2^U, ^c, \cap, \cup)$ with added operators. The lower and upper approximation operators have the following properties:

(i).	$\underline{apr}(A) = (\overline{apr}(A^c))^c$,	$\overline{apr}(A) = (\underline{apr}(A^c))^c$.
(ii).	$\underline{apr}(U) = U$,	$\overline{apr}(\emptyset) = \emptyset$,
(iii).	$\underline{apr}(\emptyset) = \emptyset$,	$\overline{apr}(U) = U$,
(iv).	$\underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B)$,	$\overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B)$.

Property (i) states that the approximation operators are dual operators with respect to set complement ^{c}. By properties (ii) and (iii), the approximations of both the universe U and the empty set $\emptyset$ are themselves. Property (iv) states that the lower approximation operator is distributive over set intersection $\cap$, and the upper approximation operator is distributive over set union $\cup$. Additional properties of approximation operators are summarized below, using the same labelling system as in modal logic [2, 18, 25, 29]:

(D).	$\underline{apr}(A) \subseteq \overline{apr}(A)$;		
(T).	$\underline{apr}(A) \subseteq A$,	(T′).	$A \subseteq \overline{apr}(A)$;
(B).	$A \subseteq \underline{apr}(\overline{apr}(A))$,	(B′).	$\overline{apr}(\underline{apr}(A)) \subseteq A$;
(4).	$\underline{apr}(A) \subseteq \underline{apr}(\underline{apr}(A))$,	(4′).	$\overline{apr}(\overline{apr}(A)) \subseteq \overline{apr}(A)$;
(5).	$\overline{apr}(A) \subseteq \underline{apr}(\overline{apr}(A))$,	(5′).	$\overline{apr}(\underline{apr}(A)) \subseteq \underline{apr}(A)$.

They follow from the definition of approximation operators.

Based on the three definitions, it is possible to investigate the connections between rough sets and other theories [18]. The element based definition relates approximation operators to the necessity and possibility operators of modal logic [25]. The granule based definition relates rough sets to granular computing [21]. The subsystem based definition relates approximation operators to interior and closure operators of topological spaces [12], and closure operators of closure systems [20]. Furthermore, they can be used to show the connection between rough set theory and belief functions [11, 26].

1.2 Algebraic methods

Algebraic methods focus on the algebraic system $(2^U, {}^c, L, H, \cap, \cup)$ without directly reference to equivalence relations, where L and H are two abstract unary operators called approximation operators [5, 15, 17]. Additional operators have been introduced to characterize the approximation operators [4, 15, 18].

The connections between constructive and algebraic methods can be established by stating axioms on L and H under which there exists an equivalence relation producing the same approximation operators [5, 16]. The main result can be stated as follows [5, 16]. Suppose L and H are a pair of dual unary operators on 2^U. There exists an equivalence relation E on U such that $L(A) = \underline{apr}_E(A)$ and $H(A) = \overline{apr}_E(A)$ if and only if L and H satisfy axioms (iv), (T), (B), and (4). The equivalence relation is defined by $[x]_E = H(\{x\})$.

Consider now three additional operators.

Upper approximation distribution: Suppose that property (iv) holds for approximation operators L and H. For any subset $A \subseteq U$ we have $H(A) = \bigcup_{x \in A} H(\{x\})$. By setting $h(x) = H(\{x\})$, we obtain an operator from the universe to the power set of the universe, namely, $h : U \longrightarrow 2^U$. By definition, this mapping is called a upper approximation distribution, and the upper approximation can be calculated by $H(A) = \bigcup_{x \in A} h(x)$. The lower approximation operator can be defined through duality. In the standard rough set model, the upper approximation operator is given by $h(x) = [x]_E$.

Basic mapping: The Boolean algebra $(2^U, ^c, \cap, \cup)$ is an atomic Boolean algebra whose atoms are singleton subsets of U. Let $\mathcal{A}(2^U)$ be the set of all atoms. The atom $\{x\}$ can be identified with the element x of U. The equivalence relation induced mapping $[\cdot]_E$ can be identified with a basic mapping, $\varphi : \mathcal{A}(2^U) \longrightarrow 2^U$. From the element based definition we have [4]:

Atom based definition:

$$L(A) = \bigcup\{a \mid a \in \mathcal{A}(2^U), \varphi(a) \subseteq A\},$$
$$H(A) = \bigcup\{a \mid a \in \mathcal{A}(2^U), \varphi(a) \cap A \neq \emptyset\}. \quad (5)$$

An advantage of this definition is that all entities under consideration are elements of the power set 2^U. Conversely, from approximation operators, we have:

$$\varphi(\{x\}) = \{y \mid y \in U, x \in H(\{y\})\}. \quad (6)$$

In the standard rough set model, the basic mapping is given by $\varphi_E(\{x\}) = [x]_E$, which is essentially the same as upper approximation distribution. If an arbitrary binary relation is used, the observation is no longer true.

Basic set assignment: Suppose a pair of approximation operators satisfy axioms (i)-(iv). One can define a mapping, $m : 2^U \longrightarrow 2^U$, called basic set assignment as follows:

$$m(A) = L(A) - \bigcup_{B \subset A} L(B). \quad (7)$$

The basic set assignment satisfies the following axioms:

$$\text{(m1).} \quad \bigcup_{A \subseteq U} m(A) = U, \qquad \text{(m2).} \quad A \neq B \Longrightarrow m(A) \cap m(B) = \emptyset.$$

The approximation operators can be obtained by:

$$L(A) = \bigcup_{B \subseteq A} m(B), \qquad H(A) = \bigcup_{B \cap A \neq \emptyset} m(B). \quad (8)$$

The connection between basic mapping and basic set assignment is given by:

$$m(A) = \{x \mid x \in U, \varphi(\{x\}) = A\}, \qquad \varphi(\{x\}) = A, x \in m(A). \quad (9)$$

In the standard rough set model, we have $m([x]_E) = [x]_E$ for equivalence classes of E, and $m(A) = \emptyset$ for all other subsets of U.

2 Generalized Rough Sets

The theory of rough sets can be generalized in several directions. Within the set-theoretic framework, generalizations of the element based definition can be obtained by using non-equivalence binary relations [9, 17, 18, 25, 29], generalizations of the granule based definition can be obtained by using coverings [9, 14, 19, 21, 30], and generalizations of subsystem based definition can be obtained by using other subsystems [20, 27]. By the fact that the system $(2^U, ^c, \cap, \cup)$ is a Boolean algebra, one can generalize rough set theory using other algebraic systems such as Boolean algebras, lattices, and posets [4, 18, 20]. Subsystem based definition and algebraic methods are useful for such generalizations.

2.1 Rough set approximations using non-equivalence relations

Let $R \subseteq U \times U$ be a binary relation on the universe, which defines a generalized approximation space $apr = (U, R)$. Given two elements $x, y \in U$, if xRy, we say that y is R-related to x, x is a predecessor of y, and y is a successor of x. For an element $x \in U$, its successor neighborhood is given by [25]:

$$R_s(x) = \{y \mid y \in U, xRy\}. \tag{10}$$

With respect to element based definition, we define a pair of lower and upper approximations by replacing the equivalence class $[x]_R$ with the successor neighborhood $R_s(x)$:

$$\begin{aligned} \underline{apr}_R(A) &= \{x \mid R_s(x) \subseteq A\}, \\ \overline{apr}_R(A) &= \{x \mid R_s(x) \cap A \neq \emptyset\}. \end{aligned} \tag{11}$$

The basic mapping is given by $\varphi(\{x\}) = R_s(x)$ and the basic set assignment is given by $m(A) = \{x \mid R_s(x) = A\}$.

The connection between the constructive and algebraic methods with respect to non-equivalence relations can be stated as follows [16, 18]. Suppose L and H are a pair of dual operators satisfying axioms (i)-(iv), there exists a serial, a reflexive, a symmetric, a transitive and an Euclidean binary relation, respectively, on U such that $L(A) = \underline{apr}_R(A)$ and $H(A) = \overline{apr}_R(A)$ if and only if L and H satisfy axioms (D), (T), (B), (4) and (5), respectively. The binary relation is defined by $R_s(x) = \{y \mid x \in H(\{y\})\}$.

2.2 Rough set approximations using coverings

A covering of a universe U is a family of subsets of the universe such that their union is the universe. By allowing nonempty overlap of two subsets, a covering is a natural generalization of a partition. The granule based definition can be used to generalize approximation operators.

Let $\mathbf{C}$ be a covering of the universe U. By replacing U/E with $\mathbf{C}$ and equivalence classes with subsets in $\mathbf{C}$ in the granule based definition, one immediately obtains a pair of approximation operators [30]. However, they are not a pair of dual operators. To resolve this problem, one may extend granule based definition in two ways. Either the lower or the upper approximation operator is extended, and the other one is defined by duality. The results are two pairs of dual approximation operators [19]:

$$\begin{aligned} \underline{apr}'_{\mathbf{C}}(A) &= \bigcup\{X \mid X \in \mathbf{C}, X \subseteq A\} \\ &= \{x \mid x \in U, \exists X \in \mathbf{C}[x \in X, X \subseteq A]\}, \\ \overline{apr}'_{\mathbf{C}}(A) &= (\underline{apr}'_{\mathbf{C}}(A^c))^c \\ &= \{x \mid x \in U, \forall X \in \mathbf{C}[x \in X \Longrightarrow X \cap A \neq \emptyset]\}, \end{aligned}$$

$$
\begin{aligned}
\underline{apr}''_{\mathbf{C}}(A) &= (\overline{apr}'_{\mathbf{C}}(A^c))^c \\
&= \{x \mid x \in U, \forall X \in \mathbf{C}[x \in X \Longrightarrow X \subseteq A]\}, \\
\overline{apr}''_{\mathbf{C}}(A) &= \bigcup\{X \mid X \in \mathbf{C}, X \cap A \neq \emptyset\} \\
&= \{x \mid x \in U, \exists X \in \mathbf{C}[x \in X, X \cap A \neq \emptyset]\}.
\end{aligned}
$$

In general, the above two approximation operators are different. More specifically, $(\underline{apr}'_{\mathbf{C}}, \overline{apr}'_{\mathbf{C}})$ satisfies axioms (i)-(iii), (T), and (4); $(\underline{apr}''_{\mathbf{C}}, \overline{apr}''_{\mathbf{C}})$ satisfies axioms (i)-(iv), and (B).

Given a reflexive binary relation R, the family of successor neighborhoods induces a covering of the universe denoted by $U/R = \{R_s(x) \mid x \in U\}$. Approximation operators defined by using U/R and R are different.

2.3 Rough set approximations using subsystems

In the standard rough set model, the same subsystem is used to define lower and upper approximation operators. When generalizing the subsystem based definition, one may use two subsystems, one for the lower approximation operator and the other for the upper approximation operator.

Rough set approximations in topological spaces

Let $(U, \mathrm{O}(U))$ be a topological space, where $\mathrm{O}(U) \subseteq 2^U$ is a family of subsets of U called open sets. The family of open sets contains $\emptyset$ and U, and is closed under union and finite intersection. The family of all closed sets $\mathrm{C}(U) = \{\neg X \mid X \in \mathrm{O}(U)\}$ contains $\emptyset$ and U, and is closed under intersection and finite union. A pair of generalized approximation operators can be defined by replacing U/E with $\mathrm{O}(U)$ for lower approximation operator, and U/E with $\mathrm{C}(U)$ for upper approximation operator. In this case, the approximation operators are in fact the topological interior and closure operators, characterized by axioms (i), (ii), (iv), (T) and (4).

Rough set approximations in closure systems

A family $\mathcal{C}(U)$ of subsets of U is called a closure system if it contains U and is closed under intersection [3]. By collecting the complements of members of $\mathcal{C}(U)$, we obtain another system $\mathcal{O}(U) = \{\neg X \mid X \in \mathcal{C}(U)\}$. According to properties of $\mathcal{C}(U)$, the system $\mathcal{O}(U)$ contains the empty set $\emptyset$ and is closed under union. We define a pair of approximation operators in a closure system by replacing U/E with $\mathcal{O}(U)$ for lower approximation operator, and U/E with $\mathcal{C}(U)$ for upper approximation operator. They satisfy axioms (iii), (T) and (4).

Rough set approximations in Boolean algebras, lattices, and Posets

Recall that $(2^U, {}^c, \cap, \cup)$ is a Boolean algebra, and $\sigma(U/E)$ is a sub-Boolean algebra. One can immediately generalize rough set theory to a Boolean algebra $(\mathbf{B}, \neg, \wedge, \vee)$ by using subsystem based definition. In this case, we can replace U with the maximum element $\mathbf{1}$, $\emptyset$ with the minimum element $\mathbf{0}$, set complement ^{c} with Boolean algebra complement $\neg$, set intersection $\cap$ with meet $\wedge$, and set union $\cup$ with join $\vee$. The resulting algebras is known as Boolean algebras with added operators [10, 16, 18]. In particular, one can define different subsystems corresponding to the previously discussed standard rough set model, topological rough set model, and closure system rough set model [20].

In an atomic Boolean algebra, one can also generalize rough set theory by using the atom based definition through the basic mapping φ. By imposing different axioms on φ, one can derive various Boolean algebras with added operators [4].

One may further generalize rough set theory by using lattices and posets [1, 4, 20]. The crucial point is the design of a subsystem which makes the subsystem based definition applicable.

3 Concluding Remarks

We discuss research results and directions about generalizing the theory of rough sets. The theory is developed using both constructive and algebraic methods, and their connections are established. In the constructive framework, three definitions of approximation operators are examined, the element based, the granule based, and the subsystem based definitions. The element based definition enables us to generalize the theory with non-equivalence relations. The granule based definition can be used to generalize the theory with coverings. The subsystem based definition can be used to generalize the theory in many algebraic systems. In comparison, algebraic methods are more applicable and can be used to generalize the theory in a unified manner.

We restrict our discussion to the operator oriented view of rough set theory by treating lower and upper approximation as a pair of unary set-theoretic operators. There are many other views of the rough set theory [16]. Many important generalizations of the theory, such as probabilistic and decision theoretic rough sets [13, 22, 23, 28], and rough membership functions [8, 24], although not mentioned in this paper, need further investigation.

References

1. Cattaneo, G. Abstract approximation spaces for rough theories, in: *Rough Sets in Knowledge Discovery*, Polkowski, L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, 59-98, 1998.
2. Chellas, B.F. *Modal Logic: An Introduction*, Cambridge University Press, Cambridge, 1980.
3. Cohn, P.M. *Universal Algebra*, Harper and Row Publishers, New York, 1965.
4. Järvinen, J. On the structure of rough approximations, *Proceedings of the Third International Conference on Rough Sets and Current Trends in Computing, LNAI 2475*, 123-130, 2002.
5. Lin, T.Y. and Liu, Q. Rough approximate operators: axiomatic rough set theory, in *Rough Sets, Fuzzy Sets and Knowledge Discovery*, W.P. Ziarko (Ed.), Springer-Verlag, London, 256-260, 1994.
6. Pawlak, Z. Rough sets, *International Journal of Computer and Information Sciences*, **11**, 341-356, 1982.

7. Pawlak, Z. *Rough Sets: Theoretical Aspects of Reasoning about Data*, Kluwer Academic Publishers, Boston, 1991.
8. Pawlak, Z. and Skowron, A. Rough membership functions, in: *Advances in the Dempster-Shafer Theory of Evidence*, Yager, R.R., Fedrizzi, M. and Kacprzyk, J. (Eds.), John Wiley and Sons, New York, 251-271, 1994.
9. Pomykala, J.A. Approximation operations in approximation space, *Bulletin of Polish Academy of Sciences, Mathematics*, **35**, 653-662, 1987.
10. Rasiowa, H. *An Algebraic Approach to Non-classical Logics*, North-Holland, Amsterdam, 1974.
11. Skowron, A. and Grzymala-Busse, J. From rough set theory to evidence theory, in: *Advances in the Dempster-Shafer Theory of Evidence*, Yager, R.R., Fedrizzi, M. and Kacprzyk, J. (Eds.), Wiley, New York, 193-236, 1994.
12. Wiweger, A. On topological rough sets, *Bulletin of the Polish Academy of Sciences, Mathematics*, **37**, 89-93, 1989.
13. Wong, S.K.M. and Ziarko, W. Comparison of the probabilistic approximate classification and the fuzzy set model, *Fuzzy Sets and Systems*, **21**, 357-362, 1987.
14. Wybraniec-Skardowska, U. On a generalization of approximation space, *Bulletin of the Polish Academy of Sciences, Mathematics*, **37**, 51-61, 1989.
15. Wybraniec-Skardowska, U. Unit Operations, *ZeszytyNaukowe Wyzszej Szkoly Pedagogicznej im Powstancow Slaskich w Opolu, Matematyka*, **XXVII**, 113-129, 1992.
16. Yao, Y.Y. Two views of the theory of rough sets in finite universes, *International Journal of Approximation Reasoning*, **15**, 291-317, 1996.
17. Yao, Y.Y. Constructive and algebraic methods of the theory of rough sets, *Information Sciences*, **109**, 21-47, 1998.
18. Yao, Y.Y. Generalized rough set models, in: *Rough Sets in Knowledge Discovery*, Polkowski, L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, 286-318, 1998.
19. Yao, Y.Y. Relational interpretations of neighborhood operators and rough set approximation operators, *Information Sciences*, **111**, 239-259, 1998.
20. Yao, Y.Y. On generalizing Pawlak approximation operators, *Proceedings of the First International Conference, RSCTC'98, LNAI 1424*, 298-307, 1998.
21. Yao, Y.Y. Information granulation and rough set approximation, *International Journal of Intelligent Systems*, **16**, 87-104, 2001.
22. Yao, Y.Y. Information granulation and approximation in a decision-theoretic model of rough sets, manuscript, 2002.
23. Yao, Y.Y. Probabilistic approaches to rough sets, manuscript, 2002.
24. Yao, Y.Y. Semantics of fuzzy sets in rough set theory, manuscript, 2002.
25. Yao, Y.Y. and Lin, T.Y. Generalization of rough sets using modal logic, *Intelligent Automation and Soft Computing, An International Journal*, **2**, 103-120, 1996.
26. Yao, Y.Y. and Lingras, P.J. Interpretations of belief functions in the theory of rough sets, *Information Sciences*, **104**, 81-106, 1998.
27. Yao, Y.Y. and Wang, T. On rough relations: an alternative formulation, *Proceedings of The Seventh International Workshop on Rough Sets, Fuzzy Sets, Data Mining, and Granular-Soft Computing, LNAI 1711*, 82-90, 1999.
28. Yao, Y.Y. and Wong, S.K.M. A decision theoretic framework for approximating concepts, *International Journal of Man-machine Studies*, **37**, 793-809, 1992.
29. Yao, Y.Y., Wong, S.K.M. and Lin, T.Y. A review of rough set models, in: *Rough Sets and Data Mining: Analysis for Imprecise Data*, Lin, T.Y. and Cercone, N. (Eds.), Kluwer Academic Publishers, Boston, 47-75, 1997.
30. Zakowski, W. Approximations in the space (U, Π), *Demonstratio Mathematica*, **XVI**, 761-769, 1983.